When must one strengthen one's induction hypothesis?

Eric Johannesson and Anders Lundstedt

Stockholm University

ECAP 9, Munich, August 22, 2017

Introduction

- Sometimes, in order to a prove an arithmetical fact $\forall x \varphi(x)$ by induction, straightforward induction "does not work" and instead one "must" use a "stronger" induction hypothesis $\psi(x)$ and prove $\forall x \psi(x)$, from which $\forall x \varphi(x)$ may be derived.
- Suppose we want to prove that, for all natural numbers n, the sum of the first n odd numbers is a perfect square.
- Straightforward induction yields the following:
  1. Base case: the sum of the first 0 odd numbers is 0, which is a perfect square.
  2. Inductive step: if the sum of the first n odd numbers is a perfect square k^2, then the sum of the first $n+1$ odd numbers is $k^2 + 2n + 1$. But it is not true that $k^2 + 2n + 1$ is a perfect square for all k and n. So we are stuck.

Introduction

- Instead, we need to prove the following stronger result by induction: for all natural numbers n, the sum of the first n odd numbers is n^2.
- Straightforward induction yields the following:
  1. Base case: the sum of the 0 first odd numbers is 0, which is 0^2.
  2. Inductive step: if the sum of the first n odd numbers is n^2, then the sum of the first $n+1$ odd numbers is $n^2 + 2n + 1 = (n+1)^2$.
- But how can it be “easier” to prove a stronger fact?
- Is the phenomenon real? Do we, in the above case for instance, really “need” to strengthen the induction hypothesis?

Formal characterization

- Here's how *not* to characterize the situation: there are formulas $\varphi(x)$ and $\psi(x)$ such that
  1. $PA \nvdash \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x'))$.
  2. $PA \vdash \psi(0) \wedge \forall x(\psi(x) \rightarrow \psi(x'))$.
  3. $PA \vdash \forall x \psi(x) \rightarrow \forall x \varphi(x)$.
- This situation is impossible. 2 implies $PA \vdash \forall x \psi(x)$, which by 3 yields $PA \vdash \forall x \varphi(x)$, which by pure logic yields $PA \vdash \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x'))$, which contradicts 1.

Formal characterization

- Instead, the situation may perhaps be characterized as follows: starting from the axioms of Peano arithmetic minus the induction axioms, we successively prove more and more theorems using logic and the rule of induction:

$$\frac{\varphi(0) \qquad \forall x(\varphi(x) \to \varphi(x'))}{\forall x \varphi(x)}$$

- Suppose that, at some stage in this process of mathematical inquiry, we have reached a theory T consisting of the axioms and hitherto proved theorems.
- Then, as we will show, the following situation may indeed arise:
  1. $T \nvdash \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x'))$.
  2. $T \vdash \psi(0) \wedge \forall x(\psi(x) \to \psi(x'))$.
  3. $T \vdash \forall x \psi(x) \to \forall x \varphi(x)$.

A minimal example

Consider the following non-standard model of Robinson arithmetic. Let $A = \{\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots\}$ be a countably infinite set disjoint from the natural numbers, and let the domain of the model be $\mathbb{N} \cup A$. Let the constant 0 be interpreted as the number 0, and extend the interpretation of the function symbols $', +, \cdot$ as follows:

1. $a_z' = a_{z+1}$ for $z \in \mathbb{Z}$.
2. $a_z + n = n + a_z = a_{z+n}$ for $z \in \mathbb{Z}$ and $n \in \mathbb{N}$.
3. $a_z + a_u = a_u + a_z = a_{z+u}$ for $z, u \in \mathbb{Z}$.
4. $a_z \cdot 0 = 0 \cdot a_z = 0$ for $z \in \mathbb{Z}$.
5. $a_z \cdot n = n \cdot a_z = a_{z \cdot n}$ for $z \in \mathbb{Z}$ and $n \in \mathbb{N} - \{0\}$.
6. $a_z \cdot a_u = a_u \cdot a_z = a_{z \cdot u}$ for $z, u \in \mathbb{Z}$.

A minimal example

In order to very that 1-3 above are possible, i.e. that there are T, φ and ψ such that

1. $T \nvdash \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x'))$.
2. $T \vdash \psi(0) \wedge \forall x(\psi(x) \rightarrow \psi(x'))$.
3. $T \vdash \forall x \psi(x) \rightarrow \forall x \varphi(x)$.

it will suffice to find true formulas $\varphi(x)$ and $\psi(x)$ (true in the sense of being satisfied by all natural numbers in the standard model as well as the non-standard model) with the following profile:

	...	a_{-2}	a_{-1}	a_0	a_1	a_2	...
$\varphi(x)$	...	0	0	1	0	0	...
$\psi(x)$	...	0	0	0	0	0	...

A minimal example

- If such formulas can be found, let T be the theory you get by adding $\psi(0) \wedge \forall x(\psi(x) \rightarrow \psi(x'))$ and $\forall x \psi(x) \rightarrow \forall x \varphi(x)$ to Robinson arithmetic. Since these sentences are true in our non-standard model, this is a model of T, verifying 1-3 above.
- For instance, let

$$\varphi(x) := \forall y \forall z(x \neq x \cdot x \wedge y + x = z + x \rightarrow y = z)$$
$$\psi(x) := \forall y \forall z(y + x = z + x \rightarrow y = z)$$

- Then we actually have the following situation (with Q being Robinson arithmetic):
  1. $Q \vdash \varphi(0)$.
  2. $Q \nvdash \forall x(\varphi(x) \rightarrow \varphi(x'))$.
  3. $Q \vdash \psi(0) \wedge \forall x(\psi(x) \rightarrow \psi(x'))$.
  4. $\vdash \forall x(\psi(x) \rightarrow \varphi(x))$.

Proof by independent or weaker induction hypothesis

Is either of the following possible?

- Proof by independent induction hypothesis:
  1. $T \nvdash \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x'))$.
  2. $T \vdash \psi(0) \wedge \forall x(\psi(x) \rightarrow \psi(x'))$.
  3. $T \vdash \forall x \psi(x) \leftrightarrow \forall x \varphi(x)$.
  4. $T \nvdash \forall x(\psi(x) \rightarrow \varphi(x))$.
  5. $T \nvdash \forall x(\varphi(x) \rightarrow \psi(x))$.
- Proof by weaker induction hypothesis:
  1. $T \nvdash \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x'))$.
  2. $T \vdash \psi(0) \wedge \forall x(\psi(x) \rightarrow \psi(x'))$.
  3. $T \vdash \forall x \psi(x) \leftrightarrow \forall x \varphi(x)$.
  4. $T \nvdash \forall x(\psi(x) \rightarrow \varphi(x))$.
  5. $T \vdash \forall x(\varphi(x) \rightarrow \psi(x))$.

Proof by independent induction hypothesis

It suffices to find true formulas $\varphi(x)$ and $\psi(x)$ with the following profile:

	...	a_{-2}	a_{-1}	a_0	a_1	a_2	...
$\varphi(x)$	...	1	1	1	0	0	...
$\psi(x)$	...	0	0	1	1	1	...

For instance, with $x < y := \exists z(x + z = y) \land x \neq y$, let

$$\varphi(x) := \forall y(y < x \to x^2 \neq y^2)$$
$$\psi(x) := \forall y(x < y \to x^2 \neq y^2)$$

Observe that $\vdash \forall x \varphi(x) \leftrightarrow \forall x \psi(x)$, by simple relabeling of variables. Thus, let $T = Q + \psi(0) \land \forall x(\psi(x) \to \psi(x'))$.

Proof by weakened induction hypothesis

It suffices to find true formulas $\varphi(x)$ and $\psi(x)$ with the following profile:

	...	a_{-2}	a_{-1}	a_0	a_1	a_2	...
$\varphi(x)$	...	0	0	1	0	0	...
$\psi(x)$	...	0	0	1	1	1	...

For instance, let

$$\varphi(x) := \forall y(x \neq y \rightarrow x^2 \neq y^2)$$
$$\psi(x) := \forall y(x < y \rightarrow x^2 \neq y^2)$$

The original example

- Let the function $f : \mathbb{N} \to \mathbb{N}$ be defined recursively as follows:

$$f(0) = 0$$
$$f(n+1) = f(n) + 2n + 1$$

- Let *minimal arithmetic* (MA) be the theory you get by adding commutativity, associativity and distribution laws for addition and multiplications to the axioms of Robinson arithmetic.
- Suppose that, using minimal arithmetic and the rule of induction, we want to show that, for any natural number n, there's a natural number k such that $f(n) = k^2$.

The original example

- In order to do that, the normal thing to do is to extend our language with a new 1-place function symbol $\mathtt{f}$, the intended interpretation of which is f, and add the following two axioms to our theory of minimal arithmetic:

  (A1) $\mathtt{f}(0) = 0$.

  (A2) $\forall x(\mathtt{f}(x') = \mathtt{f}(x) + (0'' \cdot x)')$.

- Let

$$\varphi(x) := \exists y(\mathtt{f}(x) = y \cdot y)$$

$$\psi(x) := \mathtt{f}(x) = x \cdot x$$

The original example

- It's easy to show that the non-standard model of Robinson arithmetic introduced earlier also is a model of minimal arithmetic.
- Extend this model with an interpretation $g : \mathbb{N} \cup A \to \mathbb{N} \cup A$ of `f`, defined as follows:
  1. $g(0) = 0$.
  2. $g(n + 1) = g(n) + 2 \cdot n + 1$ for $n \in \mathbb{N}$.
  3. $g(a_0) = a_1$.
  4. $g(a_{n+1}) = g(a_n) + 2 \cdot a_n + 1$ for $n \in \mathbb{N}$.
  5. $g(a_{n-1}) = g(a_n) + 2 \cdot a_n + 1$ for $n \in \mathbb{Z} - (\mathbb{N} - \{0\})$.
- The result $\mathcal{M}$ is a model of $MA + A1 + A2$.

The original example

- Moreover, we have

$$\mathcal{M}' \not\models \forall x(\exists y(\mathtt{f}(x) = y \cdot y) \to \exists y(\mathtt{f}(x') = y \cdot y))$$

as witnessed by a_0 assigned to x, and

$$\mathcal{M}' \models \forall x(\mathtt{f}(x) = x' \cdot x' \to \mathtt{f}(x') = x'' \cdot x'')$$

since $g(a_n) > a_{(n+1)^2}$ for all $n \in \mathbb{Z}$.

- With $T = MA + A1 + A2$, we thus have

$$T \not\vdash \forall x(\exists y(\mathtt{f}(x) = y \cdot y) \to \exists y(\mathtt{f}(x') = y \cdot y))$$

and

$$T \vdash \mathtt{f}(0) = 0 \cdot 0 \wedge \forall x(\mathtt{f}(x) = x \cdot x \to \mathtt{f}(x') = x' \cdot x')$$

as desired.

Thank you!