On the rationality of buying lottery tickets

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1 Introduction

Many of us sometimes buy lottery tickets. That means we are willing to bet a relatively small amount of money for a very small chance of gaining a fortune. Most of us are not willing, however, to bet all our savings for a 50% chance of doubling them. Assuming said bets are *fair* (the cost of entering them equals the monetary reward multiplied by the probability of winning), is such behavior consistent with the von Neumann-Morgenstern theory of rationality? Yes.

2 Accepting a fair bet

According to the von Neumann-Morgenstern theory of rationality, your preferences over bets can be represented by a utility function $U : \mathbb{R} \to \mathbb{R}$ over wealth. At any given level of wealth w, and assuming that the alternative is to buy no bet at all, you are willing to buy a bet with reward r at odds $p/(1-p)$ for a price $p \cdot r$ just in case

$$U(w) \leq pU(w - pr + r) + (1-p)U(w - pr)$$

Assuming that $pr > 0$, this is equivalent to

$$\frac{U(w) - U(w - pr)}{w - (w - pr)} \leq \frac{U(w - pr + r) - U(w - pr)}{(w - pr + r) - (w - pr)}$$

which means that the slope of the straight line going through $\langle w - pr, U(w - pr)\rangle$ and $\langle w, U(w)\rangle$ is less than or equal to the slope of the straight line going through $\langle w - pr, U(w - pr)\rangle$ and $\langle w - pr + r, U(w - pr + r)\rangle$. In other words: you are willing to buy the bet just in case the straight line going through $\langle w - pr, U(w - pr)\rangle$ and $\langle w, U(w)\rangle$ does not pass above $\langle w - pr + r, U(w - pr + r)\rangle$. Thus, with the straight line $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{U(w) - U(w - pr)}{w - (w - pr)}(x - (w - pr)) + U(w - pr)$$

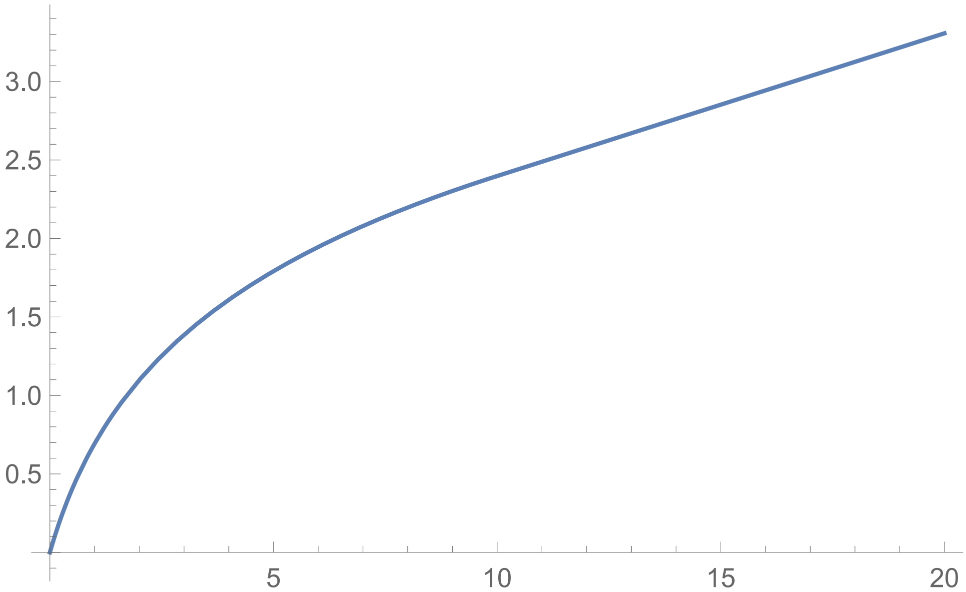
we have $f(w - pr) = U(w - pr)$ and $f(w) = U(w)$, which means that you will be willing to buy the bet just in case

$$f(w - pr + r) \leq U(w - pr + r)$$

3 A simple model

Let the utility function $U : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined by

$$U(x) = \begin{cases} \ln(x+1) & \text{if } 0 \leq x < 10 \\ \frac{x}{10+1} - \frac{10}{10+1} + \ln(10+1) & \text{if } 10 \leq x \end{cases}$$



Consider a fair bet whose price pr equals your wealth. You are willing to buy this bet just in case

$$U(pr) \leq pU(r) + (1-p)U(0)$$

which is equivalent to

$$U(pr) \leq pU(r)$$

Equality holds when $pr = 0$ or $p = 1$. Assume that $p < 1$. First we show that, when $pr < 10$, the left hand side grows faster with respect to r:

$$\frac{\partial U(pr)}{\partial r} > \frac{\partial pU(r)}{\partial r}$$

When $0 \leq r < 10$, this is equivalent to

$$\frac{\partial \ln(pr+1)}{\partial r} = \frac{p}{pr+1} > \frac{\partial p \ln(r+1)}{\partial r} = \frac{p}{r+1}$$

which is indeed the case. When $10 \leq r$, the same thing is equivalent to

$$\frac{\partial \ln(pr+1)}{\partial r} = \frac{p}{pr+1} > \frac{\partial p(\frac{r+1}{10+1} - \frac{10}{10+1} + \ln(10+1))}{\partial r} = \frac{p}{10+1}$$

which is also the case. Finally, we observe that, when $pr \geq 10$, we have

$$U(pr) > pU(r)$$

iff

$$\frac{pr+1}{10+1} - \frac{10}{10+1} + \ln(10+1) > p(\frac{r+1}{10+1} - \frac{10}{10+1} + \ln(10+1))$$

iff

$$\frac{pr}{10+1} - \frac{10-1}{10+1} + \ln(10+1) > \frac{pr}{10+1} - p\frac{10-1}{10+1} + p\ln(10+1)$$

iff

$$(1-p)\ln(10+1) > (1-p)\frac{10-1}{10+1}$$

iff

$$\ln(10+1) > \frac{10-1}{10+1}$$

iff

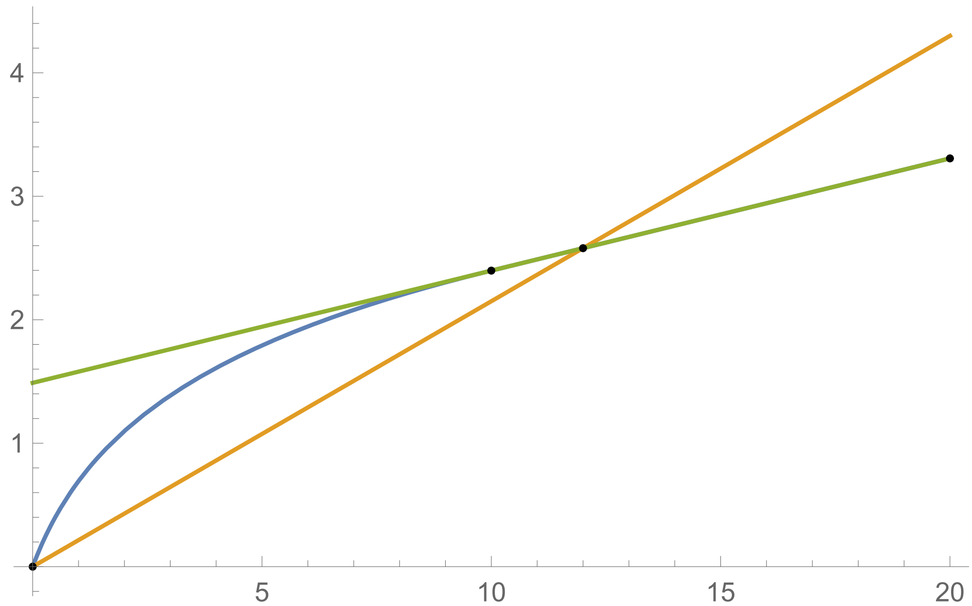
$$10+1 > e^{\frac{10-1}{10+1}}$$

which is true. Hence, whenever $0 < p < 1$ and $0 < r$, we have

$$U(pr) > pU(r) + (1-p)U(0)$$

Nevertheless, whenever $w - pr \geq 10$, we also have

$$\begin{aligned} &pU(w-pr+r) + (1-p)U(w-pr) = \\ &= p(\frac{w-pr+r}{10+1} - \frac{10}{10+1} + \ln(10+1)) + (1-p)(\frac{w-pr}{10+1} - \frac{10}{10+1} + \ln(10+1)) = \\ &= \frac{w-pr}{10+1} + p\frac{r}{10+1} - \frac{10}{10+1} + \ln(10+1) = \frac{w}{10+1} - \frac{10}{10+1} + \ln(10+1) = \\ &= U(w) \end{aligned}$$



At wealth 12, you are willing to pay 2 for a 25% chance of gaining 8. But you are not willing to pay more than 2 for a fair bet with $p < 1$. Moreover, no matter how rich or poor, you are never willing to risk all your savings on such a bet.