# Primitive recursion and tail recursion

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**Definition 1** (Basic functions)**.** The *basic functions* are the zero function $Z$, the successor function $s$, and the identity functions $id_i^n$ for any $1 \leq i \leq n$, defined by

$$\begin{aligned} Z(x) &= 0 \\ s(x) &= x + 1 \\ id_i^n(x_1, ..., x_n) &= x_i \end{aligned}$$

**Definition 2** (Composition)**.** Let $f$ be an $n$-place function and $g_1, ..., g_n$ be $m$-place functions. Define the $m$-place function $h$ by

$$h(\bar{x}) = f(g_1(\bar{x}), ..., g_n(\bar{x}))$$

Then we say that $h$ is obtained from $f$ and $g_1, ..., g_n$ by *composition*.

**Definition 3** (Primitive recursion)**.** Let $f$ and $g$ be $n$-place and $(n+1)$-place functions, respectively. Define the $(n+1)$-place function $h$ by

$$\begin{aligned} h(\bar{x}, 0) &= f(\bar{x}) \\ h(\bar{x}, y + 1) &= g(\bar{x}, y, h(\bar{x}, y)) \end{aligned}$$

Then we say that $h$ is obtained from $f$ och $g$ by *primitive recursion*.

**Definition 4** (Tail recursion)**.** Let $f$ and $g_1, ..., g_n$ be $n$-place functions. Define the $(n+1)$-place function $h$ by

$$\begin{aligned} h(\bar{x}, 0) &= f(\bar{x}) \\ h(\bar{x}, y + 1) &= h(g_1(\bar{x}), ..., g_n(\bar{x}), y) \end{aligned}$$

Then we say that $h$ is obtained from $f$ och $g$ by *tail recursion*.

**Remark 1.** We could allow $g_1, \ldots, g_n$ to be $(n+1)$-place functions taking $y$ as an additional argument. Theorem 1 below shows that it makes no difference.

**Definition 5** (Primitive recursive functions)**.** The *primitive recursive functions* are the smallest set of functions containing the basic functions that is closed under composition and primitive recursion.

**Definition 6** (Tail recursive functions)**.** The *tail recursive functions* are the smallest set of functions containing the basic functions that is closed under composition and tail recursion.

**Theorem 1.** *Every primitive recursive function is tail recursive.*

*Proof.* By induction on function descriptions. By definition, the basic functions are tail recursive, and composition preserves tail recursivity. Assume, as induction hypothesis, that $f$ and $g$ are tail recursive, and let $h$ be obtained from $f$ and $g$ by primitive recursion:

$$\begin{aligned} h(\bar{x}, 0) &= f(\bar{x}) \\ h(\bar{x}, y+1) &= g(\bar{x}, y, h(\bar{x}, y)) \end{aligned}$$

Define the tail recursive function $h'$ by

$$\begin{aligned} h'(\bar{x}, z, v, 0) &= z \\ h'(\bar{x}, z, v, y+1) &= h'(\bar{x}, g(\bar{x}, v, z), s(v), y) \end{aligned}$$

where $s$ is the successor function (a basic function). We will show that

$$h'(\bar{x}, f(\bar{x}), 0, y) = h(\bar{x}, y)$$

To so do, we first show that, for any $k$ and $\bar{x}, y, z, v$,

$$h'(\bar{x}, h(\bar{x}, v), v, y+k) = h'(\bar{x}, h(\bar{x}, v+k), v+k, y)$$

by induction on $k$. The base case $k = 0$ is obvious. Assume, as induction hypothesis, that the claim holds for $k$ (with respect to any $\bar{x}, y, z, v$). Then we get

$$\begin{aligned} h'(\bar{x}, h(\bar{x}, v), v, y+(k+1)) &= h'(\bar{x}, g(\bar{x}, v, h(\bar{x}, v)), v+1, y+(k+1)) \\ &= h'(\bar{x}, h(\bar{x}, v+1), v+1, y+k) \\ \text{[by induction hypothesis]} &= h'(\bar{x}, h(\bar{x}, (v+1)+k), (v+1)+k, y) \\ &= h'(\bar{x}, h(\bar{x}, v+(k+1)), v+(k+1), y) \end{aligned}$$

as desired. Hence, in particular,

$$h'(\bar{x}, h(\bar{x}, 0), 0, 0+k) = h'(\bar{x}, h(\bar{x}, 0+k), 0+k, 0)$$

Thus, for any $\bar{x}, y$, we have

$$h'(\bar{x}, f(\bar{x}), 0, y) = h'(\bar{x}, h(\bar{x}, y), y, 0) = h(\bar{x}, y)$$

□

**Theorem 2.** *Every tail recursive function is primitive recursive.*

*Proof.* By induction on the complexity of function description. By definition, the basic functions are primitive recursive, and composition preserves primitive recursivity. Assume, as induction hypothesis, that $f$ and $g_1, ..., g_n$ are primitive recursive, and let $h$ be obtained from $f$ and $g_1, ..., g_n$ by tail recursion:

$$\begin{aligned} h(\bar{x}, 0) &= f(\bar{x}) \\ h(\bar{x}, y+1) &= h(g_1(\bar{x}), ..., g_n(\bar{x}), y) \end{aligned}$$

For each $i \geq 1$, let $\pi(i)$ be the $i : th$ prime number. We use the fact that the $n$-place sequence coding function

$$\langle x_1, ..., x_n \rangle = \pi(1)^{x_1+1} \cdot ... \cdot \pi(n)^{x_n+1}$$

and the corresponding 2-place decoding function

$$[\langle x_1, ..., x_n \rangle]_i = x_i$$

are primitive recursive. Define the primitive recursive function $g$ by

$$\begin{aligned} g(\bar{x}, 0) &= \langle \bar{x} \rangle \\ g(\bar{x}, y+1) &= \langle g_1([g(\bar{x}, y)]_1, ..., [g(\bar{x}, y)]_n), ..., g_n([g(\bar{x}, y)]_1, ..., [g(\bar{x}, y)]_n) \rangle \end{aligned}$$

and the primitive recursive function $h'$ by

$$h'(\bar{x}, y) = f([g(\bar{x}, y)]_1, ..., [g(\bar{x}, y)]_n)$$

We will show that

$$h'(\bar{x}, y) = h(\bar{x}, y)$$

To do so, we first show that for any $k$ and $\bar{x}, y, z$,

$$h([g(\bar{x}, z)]_1, ..., [g(\bar{x}, z)]_n, y + k) = h([g(\bar{x}, z + k)]_1, ..., [g(\bar{x}, z + k)]_n, y)$$

by induction on $k$. The base case $k = 0$ is obvious. Assume, as induction hypothesis, that the claim holds for $k$ (with respect to any $\bar{x}, y, z$). We then get

$$\begin{aligned} & h([g(\bar{x}, z)]_1, ..., [g(\bar{x}, z)]_n, y + (k+1)) \\ &= h(g_1([g(\bar{x}, z)]_1, ..., [g(\bar{x}, z)]_n), ..., g_n([g(\bar{x}, z)]_1, ..., [g(\bar{x}, z)]_n), y + k) \\ \text{[by def. of } g\text{]} &= h([g(\bar{x}, z+1)]_1, ..., [g(\bar{x}, z+1)]_n, y + k) \\ \text{[by ind. hyp.]} &= h([g(\bar{x}, (z+1)+k)]_1, ..., [g(\bar{x}, (z+1)+k)]_n, y) \\ &= h([g(\bar{x}, z+(k+1))]_1, ..., [g(\bar{x}, z+(k+1))]_n, y) \end{aligned}$$

as desired. We conclude that, in particular,

$$\begin{aligned} h(\bar{x}, 0+k) &= h([g(\bar{x}, 0)]_1, ..., [g(\bar{x}, 0)]_n, 0+k) \\ &= h([g(\bar{x}, 0+k)]_1, ..., [g(\bar{x}, 0+k)]_n, 0) \\ &= f([g(\bar{x}, 0+k)]_1, ..., [g(\bar{x}, 0+k)]_n) \\ &= h'(\bar{x}, 0+k) \end{aligned}$$

and so

$$h(\bar{x}, y) = h'(\bar{x}, y)$$

□