Determinism

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September 29, 2023

Let L_T be the vocabulary for a theory of time, containing (at least) a unary predicate T (*point in time*) and a binary predicate $<$ (*earlier than*), let $L_U \supseteq L_T$ be the vocabulary for a theory of the universe, and let $L = L_U - L_T$. We will assume, for the sake of simplicity, that L only contains predicates (no constants or function symbols), all with arity greater than zero. For each $P \in L$, the intended interpretation of $P\bar{x}y$ is that the x:s are in the extension of P at time y. Consequently, for any L_U-model $\mathcal{M}$ with domain D, n-place predicate $P \in L$ and point in time $t \in T^{\mathcal{M}}$, we define the extension of P in $\mathcal{M}$ at t as

$$P^{\mathcal{M},t} := \{\langle a_1, ..., a_{n-1}\rangle \in D^{n-1} : P^{\mathcal{M}}(a_1, ..., a_{n-1}, t)\}$$

Intuitively, a theory is deterministic just in case the extensions of the theory's predicates at a particular point in time determine the extensions at every later point in time. If a theory is deterministic, does it follow that it can define the latter in terms of the former?

To answer that question, we must first specify what it means for a theory to be deterministic:

Definition 1 (Deterministic)**.** An L_U-theory Θ is *deterministic* just in case, for any L_U-models $\mathcal{M}$ and $\mathcal{M}'$ of Θ with the same domain and the same interpretation of L_T, and for any $t, t' \in T^{\mathcal{M}}$ such that $t <^{\mathcal{M}} t'$: if $P^{\mathcal{M},t} = P^{\mathcal{M}',t}$ for all $P \in L$, then $P^{\mathcal{M},t'} = P^{\mathcal{M}',t'}$ for all $P \in L$.

We also need to specify what it means for a formula to talk exclusively about extensions at a particular point in time:

Definition 2 (Restriction)**.** Let $\alpha(x)$ be an L_U-formula. For each L_U-formula φ, its *α-restriction* $[\varphi]_\alpha$ is defined recursively:

1. $[s = t]_\alpha := s = t$
2. $[P\bar{s}t]_\alpha := \begin{cases} P\bar{s}t & \text{if } P \in L_T \\ \exists x(Tx \wedge \alpha(x) \wedge P\bar{s}x) & \text{otherwise} \end{cases}$
3. $[\neg\varphi]_\alpha := \neg[\varphi]_\alpha$
4. $[\varphi \wedge \psi]_\alpha := [\varphi]_\alpha \wedge [\psi]_\alpha$
5. $[\forall\bar{x}\varphi]_\alpha := \forall\bar{x}[\varphi]_\alpha$

Whenever $\alpha(x)$ defines a unique point in time, the α-restricted formulas only talk about extensions at that point, in the following sense:

Lemma 1. *Let $\alpha(x)$ be an L_U-formula, and let $\mathcal{M}$ and $\mathcal{M}'$ be two L_U-models with the same domain and the same interpretation of L_T such that $\mathcal{M}, \mathcal{M}' \models \exists! x(Tx \wedge \alpha(x))$. Let $t, t' \in T^{\mathcal{M}}$ be the unique points in time satisfying $\alpha(x)$ in each of them, and assume that $P^{\mathcal{M},t} = P^{\mathcal{M}',t'}$ for all $P \in L$. Then, for any L_U-formula φ and assignment g, $\mathcal{M}, g \models [\varphi]_\alpha$ iff $\mathcal{M}', g \models [\varphi]_\alpha$.*

Proof. By induction on the complexity of φ. □

Using Beth's definability theorem, our question can now be answered in the positive:

Theorem 1. *Let Θ be a deterministic L_U-theory, let $\alpha(x)$ and $\beta(x)$ be L_T-formulas and φ an L_U-formula. Then there is an L_U-formula ψ such that*

$$\Theta \models \exists! x(Tx \wedge \alpha(x)) \wedge \exists! x(Tx \wedge \beta(x)) \wedge \exists x \exists y(\alpha(x) \wedge \beta(y) \wedge x < y) \rightarrow \\ \forall \bar{x}([\varphi]_\beta(\bar{x}) \leftrightarrow [\psi]_\alpha(\bar{x}))$$

Proof. Assume that Θ is a deterministic L_U-theory, and let $\alpha(x)$ and $\beta(x)$ be L_T-formulas. For each n-place predicate $P \in L$, introduce new $n-1$-place predicates P_α and P_β. Let $L_\alpha := \{P_\alpha : P \in L\}$ and $L_\beta := \{P_\beta : P \in L\}$. Let Θ' be the result of extending Θ with the following axioms for each $P \in L$:

(1) a. $\forall \bar{x}[P_\alpha \bar{x} \leftrightarrow \exists y(Ty \wedge \alpha(y) \wedge P\bar{x}y)]$
 b. $\forall \bar{x}[P_\beta \bar{x} \leftrightarrow \exists y(Ty \wedge \beta(y) \wedge P\bar{x}y)]$

Clearly, Θ' is a conservative extension of Θ with respect to L_U. Define

$$\Theta^+ := \Theta' \cup \{\exists! x(Tx \wedge \alpha(x)), \exists! x(Tx \wedge \beta(x)), \exists x \exists y(\alpha(x) \wedge \beta(y) \wedge x < y)\}$$

Let $\mathcal{M}$ and $\mathcal{M}'$ be models of Θ^+ with the same domain and the same interpretation of L_T, and let $a, b \in T^{\mathcal{M}}$ and $a', b' \in T^{\mathcal{M}'}$ be the unique points in time satisfying $\alpha(x)$ and $\beta(x)$ in $\mathcal{M}$ and $\mathcal{M}'$, respectively. Since $\alpha(x)$ and $\beta(x)$ are L_T-formulas, it follows that $a = a'$ and $b = b'$. Moreover, $a <^{\mathcal{M}} b$. Assume that $P_\alpha^{\mathcal{M}} = P_\alpha^{\mathcal{M}'}$ for all $P \in L$. By (1-a), it follows that $P^{\mathcal{M},a} = P^{\mathcal{M}',a}$ for all $P \in L$. Since $\mathcal{M}$ and $\mathcal{M}'$ are models of Θ, and Θ is deterministic, it follows that $P^{\mathcal{M},b} = P^{\mathcal{M}',b}$ for all $P \in L$. Hence, by (1-b), $P_\beta^{\mathcal{M}} = P_\beta^{\mathcal{M}'}$ for all $P \in L$. By Beth's definability theorem, there is an $L_T \cup L_\alpha$-formula π for each $P \in L$ such that

$$\Theta^+ \models \forall \bar{x}[P_\beta \bar{x} \leftrightarrow \pi(\bar{x})]$$

Clearly, for any $L_T \cup L_\alpha$-formula π, there is an L_U-formula ψ such that

$$\Theta^+ \models \forall \bar{x}(\pi(\bar{x}) \leftrightarrow [\psi]_\alpha(\bar{x}))$$

It follows that, for each L_U-formula φ, there is an L_U-formula ψ such that

$$\Theta' \models \exists! x(Tx \wedge \alpha(x)) \wedge \exists! x(Tx \wedge \beta(x)) \wedge \exists x \exists y(\alpha(x) \wedge \beta(y) \wedge x < y) \rightarrow \\ \forall \bar{x}([\varphi]_\beta(\bar{x}) \leftrightarrow [\psi]_\alpha(\bar{x}))$$

Since $[\varphi]_\beta$ and $[\psi]_\alpha$ are L_U-formulas, the desired result follows by conservativity.

□

Theorem 2. *Let Θ be a deterministic L_U-theory, and let $\alpha(x)$ and $\beta(x)$ be L_U-formulas such that*

$$\Theta \vDash \exists! x(Tx \wedge \alpha(x)) \wedge \exists! x(Tx \wedge \beta(x)) \wedge \exists x \exists y(\alpha(x) \wedge \beta(y) \wedge x < y)$$

Then, for any L_U-formula φ, there is an L_U-formula ψ such that

$$\Theta \vDash \forall \bar{x}([\varphi]_\beta(\bar{x}) \leftrightarrow [\psi]_\alpha(\bar{x}))$$

Proof. Let Θ be a deterministic L_U-theory, and let $\alpha(x)$ and $\beta(x)$ be L_U-formulas such that

(2) $\qquad \Theta \vDash \exists! x(Tx \wedge \alpha(x)) \wedge \exists! x(Tx \wedge \beta(x)) \wedge \exists x \exists y(\alpha(x) \wedge \beta(y) \wedge x < y)$

For each n-place predicate $P \in L$, introduce new $n-1$-place predicates P_α and P_β. Let $L_\alpha := \{P_\alpha : P \in L\}$ and $L_\beta := \{P_\beta : P \in L\}$. Moreover, introduce two new constants c_α and c_β, and let $L_T^+ := L_T \cup \{c_\alpha, c_\beta\}$. Let Θ^+ be the result of extending Θ with

(3) a. $\forall x(x = c_\alpha \leftrightarrow Tx \wedge \alpha(x))$
b. $\forall x(x = c_\beta \leftrightarrow Tx \wedge \beta(x))$

and the following axioms for each $P \in L$:

(4) a. $\forall \bar{x}[P_\alpha \bar{x} \leftrightarrow \exists y(Ty \wedge \alpha(y) \wedge P\bar{x}y)]$
b. $\forall \bar{x}[P_\beta \bar{x} \leftrightarrow \exists y(Ty \wedge \beta(y) \wedge P\bar{x}y)]$

Clearly, Θ^+ is a conservative extension of Θ with respect to L_U. Let $\mathcal{M}$ and $\mathcal{M}'$ be models of Θ^+ with the same domain and the same interpretation of L_T^+, and let $a, b \in T^{\mathcal{M}}$ and $a', b' \in T^{\mathcal{M}'}$ be the unique points in time satisfying $\alpha(x)$ and $\beta(x)$ in $\mathcal{M}$ and $\mathcal{M}'$, respectively. Since $c_\alpha, c_\beta \in L_T^+$, it follows by (3) that $a = a'$ and $b = b'$. Moreover, $a <^{\mathcal{M}} b$. Assume that $P_\alpha^{\mathcal{M}} = P_\alpha^{\mathcal{M}'}$ for all $P \in L$. By (4-a), it follows that $P^{\mathcal{M},a} = P^{\mathcal{M}',a}$ for all $P \in L$. Since $\mathcal{M}$ and $\mathcal{M}'$ are models of Θ, and Θ is deterministic, it follows that $P^{\mathcal{M},b} = P^{\mathcal{M}',b}$ for all $P \in L$. Hence, by (4-b), $P_\beta^{\mathcal{M}} = P_\beta^{\mathcal{M}'}$ for all $P \in L$. By Beth's definability theorem, there is an $L_T^+ \cup L_\alpha$-formula π for each $P \in L$ such that

$$\Theta^+ \vDash \forall \bar{x}[P_\beta \bar{x} \leftrightarrow \pi(\bar{x})]$$

Clearly, for any $L_T^+ \cup L_\alpha$-formula π, there is an L_U-formula ψ such that

$$\Theta^+ \vDash \forall \bar{x}(\pi(\bar{x}) \leftrightarrow [\psi]_\alpha(\bar{x}))$$

It follows that, for each L_U-formula φ, there is an L_U-formula ψ such that

$$\Theta^+ \vDash \forall \bar{x}([\varphi]_\beta(\bar{x}) \leftrightarrow [\psi]_\alpha(\bar{x}))$$

The desired result follows by conservativity. □