A curious case of empirical refutation

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I once gave my philosophy of science students the following assignment: provide an example of an unsuccessful scientific theory or hypothesis, and provide examples of empirical observations that have been taken to refute it.

As an example of an unsuccessful scientific hypothesis, one student wanted to use a conjecture of Fermat stating that, for any natural number n, the number $2^{2^n} + 1$ is prime. By way of refutation, the student cited Euler's discovery that $2^{2^5} + 1 = 641 \times 6700417$. Quite understandably, the student was not sure how to construe this as a case of *empirical* refutation, but I encouraged him to try. After all, I did advocate a Quinean epistemology of mathematics during my lectures.

Being mindful of the holistic nature of the matter (the Quine-Duhem thesis), the student noted that Fermat's conjecture, in conjunction with certain other axioms of arithmetic, logically entails a contradiction, which was essentially what Euler had discovered. Even so, the student was still not quite sure how to turn this into a case of empirical refutation. What empirical observation – if any – did Euler make that, perhaps in conjunction with various other assumptions (e.g. certain axioms of arithmetic), logically contradicted Fermat's conjecture?

Here is an easy – but perhaps not very good – answer: any observation that you like. Assuming certain axioms of arithmetic, in conjunction with Fermat's conjecture, Euler would have been able to derive any conclusion whatsoever, including the conclusion that grass is blue. The observation that grass is green would then have sufficed to empirically refute Fermat's conjecture (modulo the the Quine-Duhem thesis, of course).

The reason the answer is not a very good one is that it does not seem to correspond to what Euler actually did. Presumably, Euler did not finish his calculation, look out the window to check whether grass is blue, and then conclude that Fermat must be wrong. The only relevant empirical investigation Euler conducted, presumably, was to check that his calculation had been done correctly. Perhaps, then, this is the empirical observation we are looking for?

Let us assume, for the sake of definiteness, that Euler derived the negation of Fermat's conjecture from a finite number of axioms of arithmetic, and that he did so in a system of natural deduction. Let us also assume that he wrote the whole thing down on a piece of paper, thus creating a physical object with certain observable properties. What reasonable assumptions may we rely upon, in conjunction with Fermat's conjecture, to derive the conclusion that such an object does not exist, i.e. that there is no configuration of ink marks on paper instantiating a natural deduction proof from certain axioms of arithmetic to the negation of Fermat's conjecture?

There are two reasonable assumptions we may rely upon: one is the soundness of the rules of natural deduction (that they preserve truth), and the other is the truth of the relevant axioms of arithmetic. The latter can be obtained from the axioms themselves by semantic ascent: from P, conclude that 'P' is true. Likewise, the truth of Fermat's conjecture can be obtained by assuming Fermat's conjecture. Do these three premises – (i) the soundness of the rules, (ii) the truth of the relevant axioms, and (iii) the truth of Fermat's conjecture – now jointly entail that the physical object in question – Euler's derivation – does not exist? Not quite – we also need to assume that a sentence and its negation cannot both be true. But granted as much, it seems like we have succeeded.

However, and this is something of a conundrum: in order to derive the non-existence of Euler's proof, it suffices to assume the relevant axioms of arithmetic in conjunction with Fermat's conjecture. Since these jointly entail a contradiction, they also entail the non-existence of Euler's proof (even as a physical object). The assumption of soundness, which at first seemed so crucial to the whole enterprise, turns out to be redundant! I am not quite sure what to make of that. Perhaps somewhere in here is an argument for relevance logic (against the principle of explosion); or perhaps the lesson is simply that Euler's refutation of Fermat's conjecture cannot be construed as a case of empirical refutation.